A General Method for Flutter Optimization

L. B. GWIN*
Stanford University, Stanford, Calif.
R. F. TAYLOR†
Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio

A numerical procedure is presented for determination of optimal member sizes of aircraft structural components such that weight is minimized subject to a specified lower bound on flutter speed. The method has been devised to utilize the most general and accurate of current analytical flutter prediction methods so that substructures of arbitrary aerodynamic and structural complexity can be efficiently synthesized with substantial numbers of design parameters. Expressions for first-order flutter derivatives are developed for driving a gradient based mathematical program. Application to a sample problem of just two design variables is used to illustrate the method and the optimization of a larger problem is included to indicate favorable operational efficiency.

I. Introduction

OPTIMAL design of large-scale, realistic structures with constraints on dynamic behavior is characterized by an increase in analysis requirements in comparison with static constraint problems. If rigorous dynamic analysis is necessary to update the structure for each design iteration in an optimization algorithm, the design refinement of large problems becomes prohibitive. This is particularly true in the case of flutter optimization of aircraft substructures where the analysis loop for accurate determination of the flutter speed typically requires mass and stiffness definition, vibration analysis, conversion of discrete modal amplitudes to polynomial or spline representation, generation of generalized unsteady aerodynamic forces, flutter eigenvalue solution, and matchpoint evaluation.

In the following sections a numerical procedure is described whose objective is to determine optimal design of structures of practical scale with flutter constraints, maintaining the generality required for accurate aeroelastic analysis as well as the efficiency required for large-scale optimization. The basic compromise necessary to determine material distribution such that lifting surface structures are simultaneously lightweight and above a specified flutter margin is resolved by posing and solving a basic mathematical programing problem. Using only design parameters that size internal structural members, total weight is selected as the objective to be minimized subject to a specified lower bound on flutter speed at a fixed altitude and the necessary limitations parameter magnitudes. Although this is perhaps the fundamental of aeroelastic optimization problems, its satisfactory solution has been elusive and it forms the basis for a significant history of investigation in the literature.

The key to proper management of the time-consuming analysis problem associated with the flutter constraint is found in the suggestion of previous investigators[1] that approximate but accurate intermediate analysis be utilized during the synthesis process. It is also necessary to develop efficient updating procedures for mass and stiffness quantities for rapid constraint evaluation and to devise an effective driver for the optimization strategy. For this class of problem, feasible-direction methods, previously discarded for large static constraint applications, become attractive, especially in the absence of any established optimality criterion for nonconservative problems. It therefore becomes necessary to develop efficient calculation procedures to generate gradients to the constraint hypersurfaces, which is complicated for flutter derivatives by the lack of an explicit expression for flutter speed in terms of the design variables.

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* Research Assistant, Department of Aeronautics and Astronautics. Student Member AIAA.

† Aerospace Engineer. Member AIAA.

II. Description of the Method

Approximate flutter analysis can be made about an accurate flutter point if the assumption is made that a finite number of mode shapes of the initial design will serve as good approximations to the dynamic structural behavior of subsequent iterated designs. This permits a complete separation of the rigorous flutter analysis routines, used to accurately determine the flutter point of the initial design, from the optimization process itself. A synthesis procedure independent of the methods of analysis then has significance for the capability to interchange structural and aerodynamic computational schemes.

The generalized equations of harmonic motion for dynamic neutral stability are considered in the form required for use in the V-g method of flutter determination[2]

$$(-\omega^2[M]+(1+ig)[K]-\omega^2[Q])\{q\}=\{0\} \quad (1)$$

where $[M]$ = generalized mass matrix, $[K]$ = generalized stiffness matrix, $[Q]$ = matrix of generalized nonstationary aerodynamic forces; $\{q\}$ = vector of corresponding generalized coordinates; ω = frequency of oscillation at flutter, and g = artificial structural damping parameter.

It becomes convenient to adopt Turner's notion[3] of assembling the generalized mass and stiffness from component matrices associated with each of n design variables t_i and superimposing a possible nonactive generalized mass and stiffness matrix of subscript zero

$$[M] = [M_0] + \sum_{i=1}^{n} t_i[M_i] \quad (2)$$

$$[K] = [K_0] + \sum_{i=1}^{n} t_i[K_i] \quad (2a)$$

This formulation permits a preliminary flutter analysis of complete generality using techniques suited to the particular structure to be synthesized and to its flight environment. A substructure of arbitrary structural and aerodynamic planform complexity would typically require finite element mass and stiffness generation with perhaps substructuring and modal synthesis to establish m discrete modal amplitude vectors that would then be used to compute the corresponding generalized matrix quantities of Eqs. (2) and serve as modal definition for computation of unsteady airloads by three-dimensional linearized potential flow techniques. The authors have experienced little difficulty in modifying existing structural analysis computer code to output the component mass and stiffness matrices required to form the contributions to Eqs. (2). This information can then be stored and supplied along with necessary details from the preliminary flutter analysis as input to the optimization program itself.

A feasible direction strategy is used to minimize a linear objective function expressing structural weight

$$W(\{t\}) = \sum_{i=1}^{n} c_i t_i = \{c\}^T\{t\} \tag{3}$$

subject to the primary constraint on flutter

$$V_F(\{t\}) \geqq V_{FS} \tag{4}$$

and the n minimum-value constraints imposed for a physically meaningful solution

$$t_i \geqq T_i, \qquad i = 1,2,\ldots,n \tag{5}$$

where the c_i, V_{FS}, and T_i are known constants.

Zoutendijk's formulation of the direction-finding problem at constraint boundaries[4] involves finding a direction vector $\{S\}$ and a scalar β such that β is maximized under the following conditions:

i) $\{S\}^T\{\nabla h_j\}+\theta_j\beta \leqq 0 \quad j\varepsilon_J$
ii) $\{S\}^T\{\nabla W\}+\beta \leqq 0$
iii) the length of $\{S\}$ is bounded

where $\{\nabla h_j\}$ = gradient of the jth constraint function, $\{\nabla W\}$ = gradient of the objective function, θ_j = positive constant for adjustment of search direction, and J = that set of constraints active at the point in question.

Such an approach is useful for direct solution of constrained minimization problems when information is available on rates of change of the objective function and constraints with respect to the design variables. This method was selected because of its avoidance of large numbers of design iterations and the associated necessity of repeated analysis which is a major consideration in an aeroelastic optimization problem. Each step in the design space guarantees an improved design that reduces weight and stays to within a linear approximation inside the constraint boundaries. The first reference serves as an adequate description of feasible direction methods and details a solution procedure by Fox for the suboptimization problem of direction determination by simplex algorithm.

III. Constraint Evaluations

Feasible direction methods are distinguished by their rapid convergence on general nonlinear problems with many constraints. The sequence of design improvement steps are taken as

$$\{t\}_{k+1} = \{t\}_k + \alpha_k\{S\} = \{t\}_k + \{\Delta t\}_k \tag{6}$$

where $\{S\}$ is a carefully selected search direction, suitably normalized, and α_k is a scalar multiplier fixing step size. When such a step is made in the n-dimensional design space an evaluation is required of all constraints to determine proximity of the new design to constraint boundaries. The new design is easily checked against the inequalities (5) but evaluating inequality (4) requires determination of the new flutter speed V_F at the point $\{t\}_{k+1}$. This implies updating the generalized mass and stiffness matrices, which in discrete form would be an extensive operation. However, the stepping procedure of Eq. (6) can be effectively used to update by employing previously determined generalized mass and stiffness matrices from Eqs. (2). If step $\{\Delta t\}$ is taken, the corresponding generalized mass matrix becomes

$$\begin{aligned}[M]_{k+1} &= [M_0] + \sum_{i=1}^{n} (t_i+\Delta t_i)[M_i] \\ &= [M]_k + \sum_{i=1}^{n} \Delta t_i[M_i] \\ &= [M]_k + \sum_{i=1}^{n} [\Delta M_i]\end{aligned} \tag{7}$$

This indicates that the updated system is obtained by adding a delta matrix formed by scaling the fixed generalized component matrices by the appropriate change in the design variable Δt_i. In a similar fashion, the new generalized stiffness matrix becomes

$$[K]_{k+1} = [K]_k + \sum_{i=1}^{n} [\Delta K_i] \tag{8}$$

The third term of Eq. (1) requires no updating if the assumption of using the modes of the initial design is maintained.

The updated generalized flutter equation (1) can then be solved at two or more points in the vicinity of the V-g crossover in order to interpolate for the new flutter speed. Although this calculation involves numerical solution of complex eigenvalue problems it is extremely rapid due to the small matrix size involved. If only m modes are required to adequately represent the aeroelastic behavior of the system, then the matrices of Eq. (1) are of order m.

IV. Gradient Calculations

The solution of the direction-finding problem for feasible direction methods requires gradient vectors normal to both the constraint surfaces and the objective function contours at the design point in question.

The gradient to the linear objective function of Eq. (3) is constant at all points in the design space

$$\{\nabla W\}^T = \left[\frac{\partial W}{\partial t_1}\frac{\partial W}{\partial t_2}\cdots\frac{\partial W}{\partial t_i}\cdots\frac{\partial W}{\partial t_n}\right] = \{c\}^T \tag{9}$$

If the n minimum-value constraints (5) are expressed as linear functions h_i of the design variables t_i,

$$h_i \equiv T_i - t_i \leqq 0 \qquad i = 1,2,\ldots,n \tag{10}$$

with T_i some preselected limit value, the gradient vectors for these constraints are also everywhere constant

$$\frac{\partial h_i}{\partial t_i} = \begin{cases} 0 & i \neq j \\ -1 & i = j \end{cases} \tag{11}$$

$$\{\nabla h_i\}^T = [0\ 0\ 0\ldots -1\ldots 0\ 0] \tag{12}$$

Similarly, the flutter constraint (4) may be written

$$h_F \equiv V_{FS} - V_F \leqq 0 \tag{13}$$

where V_{FS} is the specified flutter speed taken as that of the initial design in the applications which follow. Expressed in this manner, constraint evaluations at any particular design point indicate in a quantitative manner the degree to which each constraint is satisfied. Negative evaluations show the design to be in the feasible domain of acceptable designs according to the sign conventions stated by Eqs. (10) and (13).

Since the specified flutter speed V_{FS} is constant, the components of the flutter gradient at any design point are derivatives of the flutter speed of that design

$$\partial h_F/\partial t_i = -\partial V_F/\partial t_i \tag{14}$$

$$\{\nabla h_F\}^T = -\left[\frac{\partial V_F}{\partial t_1}\frac{\partial V_F}{\partial t_2}\cdots\frac{\partial V_F}{\partial t_i}\cdots\frac{\partial V_F}{\partial t_n}\right] \tag{15}$$

Although no explicit expression exists for V_F as a function of the t_i when Eq. (1) is solved numerically, accurate flutter derivatives can be efficiently computed by considering an analytical perturbation of the original crossover curve identifying the zero damping point in the V-g plane. Even though a machine automated interpolation procedure is required to locate the flutter point just as the original point is located, the method is exact in the determination of the perturbed curve.

A linear interpolation for V_F is made from coordinates (V_1, g_1) and (V_2, g_2) which bracket the crossover point on the flutter mode curve.

$$V_F = V_1 - [(V_2 - V_1)/(g_2 - g_1)]g_1 \tag{16}$$

Direct differentiation of this expression with respect to design variable t_i yields

$$\frac{\partial V_F}{\partial t_i} = \frac{\partial V_1}{\partial t_i} - \frac{(V_2 - V_1)\dfrac{\partial g_1}{\partial t_i} + \left(\dfrac{\partial V_2}{\partial t_i} - \dfrac{\partial V_1}{\partial t_i}\right) g_1}{(g_2 - g_1)} + \frac{(V_2 - V_1)\left(\dfrac{\partial g_2}{\partial t_i} - \dfrac{\partial g_1}{\partial t_i}\right) g_1}{(g_2 - g_1)^2} \tag{17}$$

The coordinates of any point on the curve can be extracted from the complex eigenvalue at that point defined from Eq. (1) as

$$\Omega \equiv (1 + ig)/\omega^2 \tag{18}$$

Hence

$$\omega = 1/(Re\{\Omega\})^{1/2} \tag{19}$$

$$V = (b/k)1/(Re\{\Omega\})^{1/2} \tag{19a}$$

$$g = Im\{\Omega\}/Re\{\Omega\} \tag{19b}$$

where k is the reduced frequency used to generate the curve point.

Thus derivatives of V and g in Eq. (17) can be formed as derivatives of Ω.

$$\frac{\partial V_j}{\partial t_i} = -\frac{(b/2k)(\partial/\partial t_i)(Re\{\Omega_j\})}{(Re\{\Omega_j\})^{3/2}} = -\frac{(V_j/2)Re\{\partial\Omega_j/\partial t_i\}}{Re\{\Omega_j\}} \qquad j = 1,2 \tag{20}$$

$$\frac{\partial g_j}{\partial t_i} = \frac{(\partial/\partial t_i)(Im\{\Omega_j\})}{Re\{\Omega_j\}} - \frac{Im\{\Omega_j\}(\partial/\partial t_i)(Re\{\Omega_j\})}{(Re\{\Omega_j\})^2} = \frac{(Im\{\partial\Omega_j/\partial t_i\} - g_j Re\{\partial\Omega_j/\partial t_i\})}{Re\{\Omega_j\}} \qquad j = 1,2 \tag{20a}$$

In turn, derivatives of Ω are obtained through differentiation of Eq. (1). Using definition (18), the flutter eigenvalue problem (1) is stated as

$$[A]\{q\} = \{0\} \tag{21}$$

with

$$[A] \equiv \Omega[K] - [M] - [Q] \tag{21a}$$

Systematically differentiating with respect to design variable t_i produces

$$\left(\frac{\partial\Omega}{\partial t_i}[K] + \Omega\frac{\partial}{\partial t_i}[K] - \frac{\partial}{\partial t_i}[M] - \frac{\partial}{\partial t_i}[Q]\right)\{q\} + [A]\frac{\partial}{\partial t_i}\{q\} = 0 \tag{22}$$

If the flutter analysis is performed at fixed Mach number, the generalized aerodynamic matrix in this expression is dependent only on reduced frequency k and the mode shapes of natural vibration. During the optimization procedure the natural modes of the initial design are used as primitive modes for subsequent designs. Also, the derivatives of Eqs. (20) and (22) can be evaluated at preselected values of k_1 and k_2 throughout the optimization. Under these assumptions, matrix $[Q]$ is invariant and its derivative in Eq. (22) is zero.

If the adjoint of the regular eigenvalue problem (21) is defined as

$$[A]^T\{r\} = \{0\} \tag{23}$$

this system will have the same eigenvalues as the regular problem but different eigenvectors, and transposing Eq. (23) shows that

$$\{r\}^T[A] = \{0\}^T \tag{24}$$

Premultiplying Eq. (22) by the transpose of the adjoint mode $\{r\}$ corresponding to eigenvalue Ω and eigenvector $\{q\}$ then gives the following scalar equation:

$$(\partial\Omega/\partial t_i)\bar{K} + \Omega\bar{K}_i - \bar{M}_i = 0 \tag{25}$$

where

$$\bar{K} \equiv \{r\}^T[K]\{q\} \tag{26}$$

$$\bar{K}_i \equiv \{r\}^T(\partial[K]/\partial t_i)\{q\} \tag{26a}$$

$$\bar{M}_i \equiv \{r\}^T(\partial[M]/\partial t_i)\{q\} \tag{26b}$$

Solving for the desired frequency derivatives from Eq. (25) gives

$$\partial\Omega/\partial t_i = (\bar{M}_i - \Omega\bar{K}_i)/\bar{K} \tag{27}$$

Derivatives of the generalized mass and stiffness matrices required to form $\bar{K}_i$ and $\bar{M}_i$ are readily computed from Eqs. (2) and (2a)

$$(\partial/\partial t_i)[M] = [M_i] \tag{28}$$

$$(\partial/\partial t_i)[K] = [K_i] \tag{28a}$$

Because these generalized component matrices are constant, they may be computed outside the design loop.

If a linear interpolation has been used to identify the flutter speed by solving Eq. (1) at points 1 and 2 on the V-g curve bracketing the crossover in a flutter analysis prior to optimization, the right side of Eq. (17) can be formed at any arbitrary design point by computing Ω_j, V_j, g_j, $\{q\}_j$ at the same two points and then successively forming the quantities of Eqs. (26), (27) and (20).

The work of Van de Vooren[5] contains a mathematical treatment of the changes in flutter eigenvalues and eigenvectors as perturbations are made in structural parameters. Derivatives of Ω formed in this way can be shown equivalent to Eq. (27).

V. Subsonic Wing Application

To illustrate the method and help visualize the results, the simple cantilever wing of Fig. 1 was optimized using only two design parameters. Aerodynamic loading from the external shell is transmitted to a single-cell box beam of uniform circumferential thickness. Variables t_1 and t_2 represent the cell thickness of the inboard and outboard bays, respectively. Dimensions and material properties are presented in the technical report[6] that documents development of the method.

The structural box was modeled for finite element mass and stiffness generation with beam bending and torsional elements between seven nodes. Each node was thus permitted one translational and two rotational degrees of freedom with the root node completely fixed. The first six modes of a vibration analysis were then converted to polynomial form to compute generalized nonstationary aerodynamic force matrices using the doublet lattice method.[7] The flutter speed was found to be 690 fps and this value was taken as V_{FS} in the flutter constraint of Eq. (13).

The method of feasible directions was programed to incorporate the constraint evaluations of Sec. III and the gradient calculations of Sec. IV. The generalized component matrices of Eqs. (28) were formed with output from the finite element analysis and the discrete modes and were read as input data.

With the T_i zero in the minimum-value constraints of Eqs. (10) and the θ parameter of condition (i) in the direction-finding problem adjusted to keep the direction vector close to the flutter constraint boundary, the computerized procedure recorded a

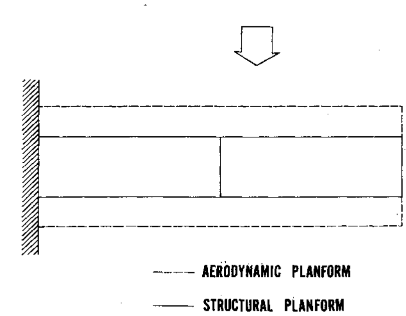

Fig. 1 Subsonic wing.

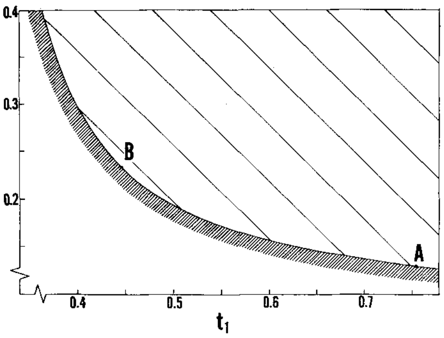

Fig. 2 Flutter constraint boundary for subsonic wing.

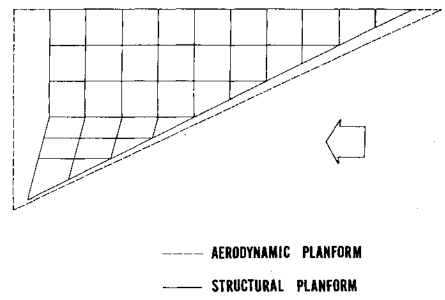

Fig. 3 Supersonic wing.

number of points at which the flutter constraint was active (to within a specified tolerance value of 0.2 fps), thus identifying its location in the two-dimensional design space. Figure 2 shows the flutter constraint boundary determined in this fashion and the objective function contours which appear as straight lines in the feasible portion of design space. Any combination of t_1 and t_2 that lies on the boundary produces a flutter speed equal to the specified value, at least to within the approximations employed in the synthesis procedure.

Point A on the curve represents the initial design and point B that of the minimum-weight design at the same flutter speed. Some details of the optimization are presented in Table 1. Other boundary points were located using trial designs and restarting the procedure at points above B.

VI. Supersonic Wing Application

The efficiency of the method was then tested by application to the delta wing of a supersonic aircraft. Figure 3 shows the planform where each of 66 nodes on the upper and lower surface were permitted three translational degrees of freedom in a finite element analysis using only square, quadrilateral, and triangular membrane elements. The 22 nodes of the wing-body intersection were completely fixed. Dimensions and material properties are specified in the technical report.[6]

The degrees of freedom in the finite element structural model were then collapsed for dynamic analysis down to motion of the upper surface normal to the planform. The first five modes of the reduced system were then represented as polynomials for airload generation using the supersonic Mach box technique.[8] A matchpoint flutter velocity of 2023 fps at Mach 2 and 18,000 ft was determined by varying altitude number, and reduced frequency.

The optimal material distribution of the wing was obtained in three separate optimizations. Cover panel thicknesses in each of the 33 planform bays were used as design variables in the first problem starting with a uniform value for each variable. The spar panel thicknesses were selected as a second problem followed by an optimization of the rib and leading edge panels. Statistical details of each run are compared in Table 1.

Figure 4 shows the changes in flutter speed and panel weight as the number of design improvement steps increased in the cover panel optimization. Normalization was made to the initial design values of $V_{FS} = 2023$ fps and $W_S = 522$ lbs.

The β parameter of conditions (i) and (ii) of the direction-finding problem is used as an indicator of proximity to an optimal solution since $\beta_{MAX} = 0$ satisfies the Kuhn-Tucker conditions necessary for a relative minimum. Stopping the optimization driver at $\beta_{MAX} \leqq 0.2$ in the cover panel problem would have produced a 25% weight reduction after only 2.7 min of computation. This corresponds to design step 30 in Fig. 4. Similar terminations for the spar and rib problems indicate most of the excess weight was removed after 1 min of computation.

VII. Conclusions

The use of approximate flutter analysis during optimization eventually introduces a drift from the true value of V_{FS}. For example, recalculation of the flutter speed using the optimal design from the cover panel problem gives a value 5% lower than that of the initial design. Since the approximation is equivalent to using a finite number of the initial modes as "primitive" modes in following approximations, the drift can be reduced by increasing the original number of modes. It is also reasonable to assume that restarting the optimization from the "optimal" design with updated flutter information would produce a new optimum with very little new drift and much the same weight saving.

The applications indicate that a gradient-based optimization scheme is effective computationally on systems as inherently complicated as the basic flutter optimization problem. The method thus represents a practical scheme for optimization of realistic structural configurations in either the subsonic or supersonic flight regimes. Efficiency can be attributed to the small matrix size possible when modal equations are used and to the excellent convergence from the optimization driver.

This approach is apparently an improvement over similar

Table 1 Optimization results

	Subsonic	Covers	Spars	Ribs
Computer	IBM 360/67	CDC 6600	CDC 6600	CDC 6600
Run time (min)	0.4	6.1	1.9	3.7
Weight reduction (%)	7	26	41	43
Steps required	24	53	17	31
Flut. grad. calculations	14	34	8	16
Direction-finding prob.	14	45	13	24
Initial β_{MAX}	0.612	7.89	10.68	9.32
Final β_{MAX}	0.003	0.06	0.07	0.08

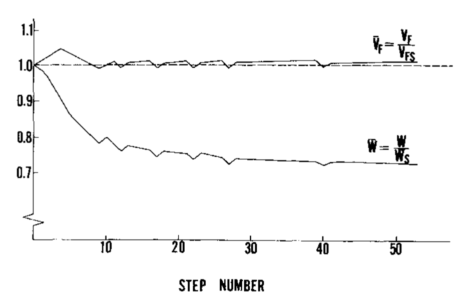

Fig. 4 Speed and weight during cover sheet optimization.

first-order computational schemes (for example Ref. 9) since derivatives of aerodynamic quantities are not required, the specified flutter speed remains relatively stable (see Fig. 4), and run times are small for the number of parameters involved. When three-dimensional aerodynamic methods are employed in the preliminary flutter analysis, particularly in conjunction with a match point determination, the method has the capability of starting with an accurately determined flutter speed on the flutter constraint hypersurface and performing design improvement steps that remain close to the specified flutter speed until a local minimum is established for the problem. The generality of the method and the computational efficiency free the user from dependence on strip and piston theory aerodynamics and limitation to predesign problems with limited numbers of design variables.

References

[1] Schmit, L. A. and Fox, R. L., *Structural Synthesis Notebook*, AIAA Seminar on Structural Optimization, AIAA Educational Programs, New York, 1970, Sec. 7.

[2] Bisplinghoff, R. L. and Ashley, H., *Principles of Aeroelasticity*, Wiley, New York, 1962, pp. 385–388.

[3] Turner, M. J., "Optimization of Structures to Satisfy Flutter Requirements," *AIAA Journal*, Vol. 7, No. 5, May 1969, pp. 945–951.

[4] Zoutendijk, G., *Methods of Feasible Directions*, Elsevier Press, Amsterdam, Holland, 1960.

[5] Van de Vooren, A. I., *Theory and Practice of Flutter Calculations for Systems with Many Degrees of Freedom*, Doctoral Thesis Published by E. Ijdo N. V., Leyden, Holland, 1952.

[6] Gwin, L. B. and McIntosh, S. C., Jr., "Large-Scale Flutter Optimization of Lifting Surfaces, Part I, Analytical Investigation," AFFDL-TR-72-22, Feb. 1973, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio.

[7] Albano, E. and Rodden, W. P., "A Doublet-Lattice Method for Calculating Lift Distributions on Oscillating Surfaces in Subsonic Flow," *AIAA Journal*, Vol. 7, No. 2, Feb. 1969, pp. 279–285.

[8] Moore, M. T. and Andrew, L. V., "Unsteady Aerodynamics for Advanced Configurations, Part IV—Application of the Supersonic Mach Box Method to Intersecting Planar Lifting Surfaces," AFFDL-TDR-64-162, Part IV, May 1965, Wright-Patterson Air Force Base, Ohio.

[9] Rudasil, C. S. and Bhatia, K. G., "Optimization of Complex Structures to Satisfy Flutter Requirements," *AIAA Journal*, Vol. 9, No. 8, Aug. 1971, pp. 1487–1491.